

# Riemann-Roch isomorphism, Chern-Simons invariant and Liouville action

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## Abstract

Using the arithmetic Schottky uniformization theory, we show the arithmeticity of  $PSL_2(\mathbb{C})$  Chern-Simons invariant. In terms of this invariant, we give an explicit formula of the Riemann-Roch isomorphism as Zograf-Mcintyre-Takhtajan's infinite product for families of algebraic curves. By this formula, we determine the unknown constant which appears in the holomorphic factorization formula of determinant of Laplacians on Riemann surfaces via the classical Liouville action. As an application, we show the rationality of Ruelle zeta values for Schottky uniformized 3-manifolds.

## 1. Introduction

Arakelov theory, Chern-Simons theory and Liouville theory have different origins and important roles in many areas of mathematics containing arithmetic (algebraic) geometry, differential geometry, global analysis and mathematical physics. The aim of this paper is to show that a combination of the 3 theories contributes to these advances. More precisely, we consider together the subjects:

- the arithmetic Riemann-Roch theorem for families of algebraic curves (cf. [D2, Fr, GS, W]),
- the  $PSL_2(\mathbb{C})$  Chern-Simons theory for hyperbolic 3-manifolds with boundary (cf. [GM, MP]),
- the holomorphic factorization formula of determinant of Laplacians on Riemann surfaces in terms of the classical Liouville action (cf. [Z1, Z2, MT]).

As a result of this consideration, we have the following results:

- arithmeticity of the power series expansion of the  $PSL_2(\mathbb{C})$  Chern-Simons invariant (cf. Theorem 5.1),
- an explicit formula of the Riemann-Roch isomorphism under the trivialization by the arithmetic  $PSL_2(\mathbb{C})$  Chern-Simons invariant and Eichler cohomology (cf. Theorem 5.5),
- determination of the unknown constant in the holomorphic factorization formula of Zograf and McIntyre-Takhtajan (cf. Theorems 5.7 and 5.8).

We will review the main idea of this paper. The arithmetic Riemann-Roch theorem is an advanced, i.e., metrized version of the Grothendieck Riemann-Roch theorem, and is especially applied to Diophantine problem. For a family of algebraic curves, these theorems gives an isometric isomorphism (up to a nonzero constant)

$$\lambda_k^{\otimes 12} \cong \kappa^{\otimes d_k}; \quad d_k := 6k^2 - 6k + 1,$$

where  $\lambda_k$  denotes the  $k$ th tautological line bundle with Quillen metric, and  $\kappa$  denotes the Deligne pairing of the relative canonical sheaf with itself. The Liouville (field) theory gives rise to the Liouville action which is a functional on the space of conformal metrics on Riemann surfaces whose critical points correspond to hyperbolic metrics. Then the holomorphic factorization formulae relate the classical Liouville action with the determinants of Laplacians in terms of Zograf-McIntyre-Takhtajan's infinite products which are extensions of Ramanujan's delta function. Notice that these determinants corresponds to the Quillen metric on  $\lambda_k$ , and as is shown by Aldrovandi [A], the classical Liouville action gives the natural metric on  $\kappa$ . Therefore, one can suppose that the arithmetic Riemann-Roch isometry is equivalent to the holomorphic factorization formula.

In order to make this equivalence more complete, we use the theories of arithmetic Schottky uniformization [I2] and of Chern-Simons line bundles [GM]. In [I2], one has a higher genus version of the Tate curve, called *generalized Tate curves*, which become Schottky uniformized Riemann surfaces over  $\mathbb{C}$  and give local coordinates on the moduli space over  $\mathbb{Z}$  of algebraic curves. Therefore, based on the  $\mathbb{Z}$ -structure of  $\lambda_k$  and  $\kappa$ , we will describe the Riemann-Roch isomorphism using the following simple fact: *if two power series over  $\mathbb{Z}$  are known to be proportional to each other and primitive (i.e., not congruent to 0 modulo any prime), then these power series are equal up to a sign.*

In [GM], it is shown that the  $PSL_2(\mathbb{C})$  Chern-Simons invariants of Schottky uniformized 3-manifolds give rise to a hermitian line bundle, called the *Chern-Simons line bundle*,  $\mathcal{L}$  which is isometrically isomorphic to  $\lambda_1^{\otimes (-6)}$  on the Schottky space. We will show that  $\mathcal{L}^{\otimes 2}$  is actually defined on the moduli space of Riemann surfaces, and that there exists an isometric isomorphism (up to a nonzero constant)

$$\lambda_1^{\otimes 12} \cong \mathcal{L}^{\otimes (-2)}.$$

By the above isomorphisms, we have an identification

$$\mathcal{L}^{\otimes 2} \cong \kappa^{\otimes (-1)}$$

as line bundles over the Deligne-Mumford compactification of the moduli space. Then by the arithmetic Schottky uniformization theory, one can modify the exponential of the  $PSL_2(\mathbb{C})$  Chern-Simons invariant such as to have a universal expression as a power series over  $\mathbb{Z}$  which we call the *arithmetic Chern-Simons invariant*. Therefore, this gives a (local) canonical trivialization of  $\kappa$ . Furthermore, we show that the natural  $\mathbb{Z}$ -structure of the Eichler cohomology group gives a canonical trivialization of  $\lambda_k$  by studying the behavior under the degeneration of a generalized Tate curve to a pointed

Tate curve. Therefore, by the holomorphic factorization formula and the above simple fact, we have:

**THEOREM 1.1** (see Theorem 5.5 for the precise statement). *Let  $F_k$  denote Zograf-McIntyre-Takhtajan's infinite product whose main part is*

$$\prod_{\{\gamma\}} \prod_{m=k}^{\infty} (1 - q_{\gamma}^m),$$

*where  $\{\gamma\}$  runs over primitive conjugacy classes in a Schottky group  $\Gamma$ , and  $q_{\gamma}$  denotes the multiplier of  $\gamma$ . Then under the trivializations of  $\kappa$  and  $\lambda_k$  by the arithmetic Chern-Simons invariant and the Eichler cohomology group respectively, the Riemann-Roch isomorphism  $\lambda_k^{\otimes 12} \xrightarrow{\sim} \kappa^{\otimes d_k}$  for families of algebraic curves is expressed as  $\pm F_k^{-12}$ .*

Using this theorem and the arithmetic Riemann-Roch theorem, we also determine the unknown constant in the holomorphic factorization formula.

We give some comments on related works. In [MP, Theorem 1.1], McIntyre and Park express the  $PSL_2(\mathbb{C})$  Chern-Simons invariants of Schottky uniformized 3-manifolds in terms of the Bergman tau function (cf. [KK1]) and Zograf's function. Since this tau function has the modular property (cf. [KK2, (3.15)]), McIntyre-Park's formula seems to represent the isomorphism  $\lambda_1^{\otimes 24} \cong \mathcal{L}^{\otimes (-4)}$ . Another relationship between the  $PSL_2(\mathbb{C})$  Chern-Simons invariants and Deligne pairings are given by Morishita and Terashima [MoT] for 3-manifolds obtained as knot complements.

As an application of the above results, we consider the rationality of special values of the Ruelle zeta function

$$Z_{\Gamma}(s) = \prod_{\{\gamma\}} (1 - |q_{\gamma}|^s)^{-1},$$

for a Schottky group  $\Gamma$  based on the fact that if  $\Gamma \subset PSL_2(\mathbb{R})$ , then Zograf-McIntyre-Takhtajan's infinite products become Selberg type zeta values. A conjecture of Deligne [D1], which is extended by Beilinson [B] and Bloch-Kato [BK], states the following: each special value of the Hasse-Weil  $L$ -function of a motive  $\mathcal{M}$  becomes the product of its transcendental part represented by the period or regulator of  $\mathcal{M}$ , and its rational part represented by the arithmetic invariant of  $\mathcal{M}$ . As its analogy for *geometric* zeta functions, we show:

**THEOREM 1.2** (see Theorem 6.2 and Corollary 6.3 for the precise statement). *Assume that  $\Gamma \subset PSL_2(\mathbb{R})$ , and that the Riemann surface uniformized by  $\Gamma$  has a model  $C_{\Gamma}$  as an algebraic curve defined over a subfield  $K$  of  $\mathbb{C}$ . Then the modified Ruelle zeta value of  $\Gamma$  at any integer  $k > 1$  belongs to*

$$\frac{\text{period of regular } (k+1)\text{-forms on } C_{\Gamma}}{\text{period of regular } k\text{-forms on } C_{\Gamma}} \cdot c(\Gamma)^{12k} \cdot K^{\times},$$

where  $c(\Gamma)$  denotes the “value” at  $\Gamma$  of Zograf’s infinite product divided by the period of regular 1-forms on  $C_\Gamma$ .

We also express this zeta value by the discriminant of  $C_\Gamma$  based on results of Saito [S], however we have no result on the rational or transcendental property of  $c(\Gamma)$ . Notice that using a result of [I3], one can construct Schottky groups  $\Gamma \subset PSL_2(\mathbb{R})$  such that  $C_\Gamma$  are hyperelliptic and defined over  $\mathbb{Q}$ .

By using the Selberg trace formula, the leading term at the central point 0 of the Ruelle  $L$  function of a hyperbolic manifold  $X$  was studied by Fried [Fri] when  $X$  is compact, and by Park [P], Sugiyama [Su1-5] and Gon-Park [GP] when  $X$  has finite volume. Especially, Sugiyama’s results are regarded as a geometric analog to Iwasawa theory and the Beilinson conjecture. In our case where  $C_\Gamma \otimes_K \mathbb{C}$  is the boundary of the hyperbolic 3-manifold  $X$  uniformized by  $\Gamma$ , the above theorem states that the rationality of the Ruelle zeta values of  $X$  at positive integers is controlled by the field of definition of its boundary.

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## 2. The Chern-Simons line bundle

**2.1.  $PSL_2(\mathbb{C})$  Chern-Simons invariant.** For a Riemannian oriented 3-manifold  $X$  with metric  $g$  and an orthonormal frame  $S$  over  $X$ , Chern and Simons [CS] introduced the Chern-Simons invariant  $CS(g, S)$  which is defined as

$$\frac{1}{16\pi^2} \int_X S^* \left( \text{Tr} \left( \omega \wedge d\omega + \frac{2}{3} \omega \wedge \omega \wedge \omega \right) \right),$$

where  $\omega$  denotes the Levi-Civita connection form for  $g$ . This complexified version, called the  $PSL_2(\mathbb{C})$  Chern-Simons invariant, was given by Yoshida [Y]. Furthermore, Guillarmou and Moroianu [GM], McIntyre and Park [MP] extended this invariant for nonclosed 3-manifolds, and they studied the relationship between the associated hermitian holomorphic line bundle and Quillen’s determinant line bundle in the Schottky setting. Recall that in [GM, Proposition 16] and [MP, 4.5], for each Schottky uniformized 3-manifold  $X$  with hyperbolic metric  $g$ , the  $PSL_2(\mathbb{C})$  Chern-Simons invariant  $CS^{PSL_2(\mathbb{C})}$  is defined and expressed as

$$CS^{PSL_2(\mathbb{C})}(g, S) = -\frac{\sqrt{-1}}{2\pi^2} \text{Vol}_R(X) + \frac{\sqrt{-1}}{2\pi} \chi(\partial X) + CS(g, S).$$

Here  $\text{Vol}_R(X)$  denotes the renormalized volume of  $X$ , and  $\chi(\partial X)$  denotes the Euler characteristic of the boundary of  $X$ .

**2.2. Schottky uniformization.** Schottky groups of rank  $g$  are free groups generated by  $\gamma_1, \dots, \gamma_g \in PSL_2(\mathbb{C})$  which map Jordan curves  $C_1, \dots, C_g \subset \mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$  to other Jordan curves  $C_{-1}, \dots, C_{-g} \subset \mathbb{P}^1(\mathbb{C})$  (with orientation reversed). Each element  $\gamma \in \Gamma - \{1\}$  is conjugated in  $PSL_2(\mathbb{C})$  to  $z \mapsto q_\gamma z$  for some  $q_\gamma \in \mathbb{C}^\times$  with  $|q_\gamma| < 1$ , called the *multiplier* of  $\gamma$ . Therefore,

$$\frac{\gamma(z) - a_\gamma}{\gamma(z) - b_\gamma} = q_\gamma \frac{z - a_\gamma}{z - b_\gamma}$$

for some element  $a_\gamma, b_\gamma$  of  $\mathbb{P}^1(\mathbb{C})$  called the *attractive*, *repulsive* fixed points of  $\gamma$  respectively. Then the discontinuity set  $\Omega_\Gamma \subset \mathbb{P}^1(\mathbb{C})$  under the action of  $\Gamma$  has a fundamental domain  $D_\Gamma$  which is given by the complement of the union of the interiors of  $C_i$  ( $i = \pm 1, \dots, \pm g$ ). The quotient space  $\Omega_\Gamma/\Gamma$  of  $\Omega_\Gamma$  by  $\Gamma$  is a (compact) Riemann surface of genus  $g$  which we denote by  $R_\Gamma$ . Furthermore, by a result of Koebe, every Riemann surface of genus  $g$  can be represented in this manner. A Schottky group  $\Gamma$  is *marked* if its free generators  $\gamma_1, \dots, \gamma_g$  are fixed, and a marked Schottky group  $(\Gamma; \gamma_1, \dots, \gamma_g)$  is *normalized* if  $a_{\gamma_1} = 0$ ,  $b_{\gamma_1} = \infty$  and  $a_{\gamma_2} = 1$ . By definition, the Schottky space  $\mathfrak{S}_g$  of degree  $g$  is the space of marked Schottky groups of rank  $g$  modulo conjugation in  $PSL_2(\mathbb{C})$  which becomes the space of normalized Schottky groups of rank  $g$  if  $g > 1$ . Then  $\mathfrak{S}_g$  is a covering space of the moduli space of Riemann surfaces of genus  $g$ .

**2.3. Chern-Simons line bundle.** We define the Chern-Simons line bundle  $\mathcal{L}_{\mathfrak{S}_g}$  over the Schottky space  $\mathfrak{S}_g$  of degree  $g$  following Freed [F], Ramadas-Singer-Weitsman [RSW] and especially Guillarmou and Moroianu [GM].

For each Schottky group  $\Gamma \subset PSL_2(\mathbb{C})$ , denote by  $X = X_\Gamma = \mathbb{H}^3/\Gamma$  the associated hyperbolic 3-manifold, where  $\mathbb{H}^3$  denotes the 3-dimensional hyperbolic space. Then the boundary of  $X$  becomes a Riemann surface  $R = R_\Gamma$  which is Schottky uniformized by  $\Gamma$ . Denote by  $F(X)$  the frame bundle of  $X$ , and by  $C_{\text{ext}}^\infty(R, *)$  the space of sections in  $C^\infty(R, *)$  which can be extended to  $\overline{X} = X \cup R$ . Let

$$c_\Gamma : C^\infty(R, F(X)) \times C_{\text{ext}}^\infty(R, SO(3)) \rightarrow \mathbb{C}$$

be the map defined as

$$c_\Gamma(\hat{S}, a) := \exp \left( \frac{2\pi\sqrt{-1}}{16\pi^2} \left( \int_R \text{Tr}(\hat{\omega} \wedge da \wedge a^{-1}) + \int_X \frac{1}{3} \text{Tr}((\tilde{a}^{-1} d\tilde{a})^3) \right) \right),$$

where  $\hat{\omega}$  is the connection form of the Levi-Civita connection of the associated metric on  $\overline{X}$  along  $R$  in the frame  $\hat{S}$ , and  $\tilde{a}$  is any smooth extension of  $a$  on  $\overline{X}$ . Then  $c_\Gamma(\hat{S}, a)$  is well-defined by [GM, Lemma 20], and is seen to satisfy the cocycle condition

$$c_\Gamma(\hat{S}, ab) = c_\Gamma(\hat{S}, a) \cdot c_\Gamma(\hat{S}a, b).$$

We define the complex vector space  $L_\Gamma$  as the space of complex-valued functions  $f$  on  $C_{\text{ext}}^\infty(R, F(X))$  which satisfy

$$f(\hat{S}a) = c_\Gamma(\hat{S}, a) \cdot f(\hat{S}) \quad (a \in C_{\text{ext}}^\infty(R, SO(3))).$$

By this condition, any element in  $L_\Gamma$  is determined by its value on any frame which can be extended to  $\overline{X}$ , and hence one can define the Chern-Simons line bundle over  $\mathfrak{S}_g$  as

$$\mathcal{L}_{\mathfrak{S}_g} := \bigsqcup_{\Gamma \in \mathfrak{S}_g} L_\Gamma.$$

Furthermore, the cocycle has absolute value 1, and hence there exists a canonical hermitian metric  $\langle \cdot, \cdot \rangle_{\text{CS}}$  on  $\mathcal{L}_{\mathfrak{S}_g}$  defined as

$$\langle f_1, f_2 \rangle_{\text{CS}} := f_1 \left( \widehat{S} \right) \overline{f_2 \left( \widehat{S} \right)}$$

for two sections  $f_1, f_2$  of  $\mathcal{L}_{\mathfrak{S}_g}$  and  $\widehat{S} \in C^\infty(R, F(X))$ .

**2.4. Isomorphism with determinant line bundle.** We review a result of [GM] on an explicit isomorphism between the Chern-Simons line bundle with connection and hermitian structure and the determinant line bundle on the Schottky space.

Let  $R$  be a Riemann surface of genus  $g > 0$ , and  $\{\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g\}$  be a set of standard generators of  $\pi_1(R, x_0)$  for some  $x_0 \in R$  satisfying

$$(\alpha_1 \beta_1 \alpha_1^{-1} \beta_1^{-1}) \cdots (\alpha_g \beta_g \alpha_g^{-1} \beta_g^{-1}) = 1.$$

Then one can take a marked Schottky group  $(\Gamma; \gamma_1, \dots, \gamma_g)$  such that  $R = R_\Gamma$  and that each  $C_k$  is homotopic to  $\alpha_k$ . Therefore, there is uniquely a basis  $\varphi_1, \dots, \varphi_g$  of holomorphic 1-forms such that  $\int_{\alpha_j} \varphi_i$  is equal to Kronecker's delta  $\delta_{ij}$ , and then the period matrix  $\tau = \left( \int_{\beta_j} \varphi_i \right)$  becomes a symmetric matrix whose imaginary part is positive definite. For each Schottky group  $\Gamma$  of rank  $g$ ,  $\{\varphi_1, \dots, \varphi_g\}$  is a basis of the space  $H^0(R_\Gamma, \Omega_{R_\Gamma})$  of holomorphic 1-forms on  $R_\Gamma$ . Therefore, the *Hodge line bundle*  $\lambda_1$  consisting of  $\bigwedge^g H^0(R_\Gamma, \Omega_{R_\Gamma})$  ( $\Gamma \in \mathfrak{S}_g$ ) becomes a holomorphic line bundle on  $\mathfrak{S}_g$  with a holomorphic canonical section

$$\varphi := \varphi_1 \wedge \cdots \wedge \varphi_g.$$

For each Riemann surface  $R_\Gamma$ , let  $h$  be the associated hyperbolic metric, and  $\det' \Delta_h$  be the (regularized) determinant of its Laplacian defined by Ray-Singer [RS]. Then the hermitian *Quillen metric* on  $\lambda_1$  (cf. [Q]) is defined as

$$\|\varphi\|_{\text{Q}}^2 := \frac{\|\varphi\|_h^2}{\det' \Delta_h} = \frac{\det \text{Im}(\tau)}{\det' \Delta_h}$$

at  $\Gamma \in \mathfrak{S}_g$ , where  $\|\varphi\|_h$  is the hermitian product on  $\bigwedge^g H^0(R_\Gamma, \Omega_{R_\Gamma})$  induced by  $h$ . Therefore, there is the unique hermitian connection  $\nabla^{\det}$  associated to the holomorphic structure on  $\lambda_1$  and the hermitian norm  $\|\varphi\|_{\text{Q}}$ .

To describe the relationship between the Chern-Simons line bundle and the determinant line bundle, we use a formula proved by Zograf [Z1, Z2] whose generalization

by McIntyre-Takhtajan [MT] will be reviewed below. Denote by  $S_L : \mathfrak{S}_g \rightarrow \mathbb{R}$  the classical Liouville action which is explicitly described by Takhtajan and Zograf [ZT] as

$$\begin{aligned} S_L &= \frac{\sqrt{-1}}{2} \int \int_{D_\Gamma} \left( \left| \frac{\partial \log \rho}{\partial z} \right|^2 + \rho \right) dz \wedge d\bar{z} \\ &\quad + \frac{\sqrt{-1}}{2} \sum_{k=2}^g \oint_{C_k} \left( \log \rho - \frac{1}{2} \log |\gamma'_k|^2 \right) \left( \frac{\gamma''_k}{\gamma'_k} dz - \frac{\overline{\gamma''_k}}{\overline{\gamma'_k}} d\bar{z} \right) \\ &\quad + 4\pi \sum_{k=2}^g \log |c(\gamma_k)|^2, \end{aligned}$$

where  $D_\Gamma, C_k \subset \mathbb{P}^1(\mathbb{C})$  are given in 2.2,  $\rho(z)|dz|^2$  denotes the pullback of the hyperbolic metric on  $R_\Gamma$  and  $c(\gamma) = c$  for  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ .

THEOREM 2.1 (Zograf [Z1, Z2]).

(1) *There exists a holomorphic function  $F_1 : \mathfrak{S}_g \rightarrow \mathbb{C}$  such that*

$$\frac{\det' \Delta_h}{\det \operatorname{Im}(\tau)} = c_g \exp \left( -\frac{S_L}{12\pi} \right) |F_1(\Gamma)|^2,$$

*where  $c_g$  is a nonzero constant depending only on  $g$ .*

(2) *If the Hausdorff dimension  $\delta_\Gamma$  of limit set of  $\Gamma$  satisfies  $\delta_\Gamma < 1$ , then  $F_1(\Gamma)$  has the following absolutely convergent product:*

$$F_1(\Gamma) = \prod_{\{\gamma\}} \prod_{m=0}^{\infty} (1 - q_\gamma^{1+m}),$$

*where  $\{\gamma\}$  runs over primitive conjugacy classes in  $\Gamma - \{1\}$ , and  $q_\gamma$  denotes the multiplier of  $\gamma$ .*

By a result of Krasnov [Kr] (see also [TT, (1.13)]), if  $\Gamma$  is a Schottky group of rank  $g$  then for  $X = \mathbb{H}^3/\Gamma$  and  $R = R_\Gamma$ ,

$$\operatorname{Vol}_R(X) = -\frac{1}{4}S_L - \frac{\pi}{2}\chi(R) = -\frac{1}{4}S_L + \pi(g-1),$$

and hence one has:

COROLLARY 2.2. *Let the notation be as above. Then*

$$\frac{\det' \Delta_h}{\det \operatorname{Im}(\tau)} = c_g \cdot e^{(1-g)/3} \exp \left( \frac{\operatorname{Vol}_R(X)}{3\pi} \right) |F_1(\Gamma)|^2.$$

THEOREM 2.3 (Guillarmou and Moroianu [GM, Theorem 43]). *There exists a connection  $\nabla^{\mathcal{L}}$  and a hermitian metric  $\|\cdot\|_{\mathcal{L}}$  on the Chern-Simons line bundle  $\mathcal{L} = \mathcal{L}_{\mathfrak{S}_g}$  on  $\mathfrak{S}_g$  such that  $\lambda_1^{\otimes 6}$  is isomorphic to  $\mathcal{L}_{\mathfrak{S}_g}^{\otimes(-1)}$  with respect to their connections and hermitian products induced from those of  $(\lambda_1, \nabla^{\det}, \|\cdot\|_Q)$  and  $(\mathcal{L}_{\mathfrak{S}_g}, \nabla^{\mathcal{L}}, \|\cdot\|_{\mathcal{L}})$  respectively. More precisely, there is an explicit isometric isomorphism of holomorphic hermitian line bundles given by*

$$\left( \sqrt{c_g \cdot e^{1-g}} \cdot F_1 \varphi \right)^{\otimes 6} \mapsto e^{-2\pi\sqrt{-1}\text{CS}^{PSL_2(\mathbb{C})}},$$

where  $F_1$  is the holomorphic function on  $\mathfrak{S}_g$ , and  $\varphi$  is the canonical section of  $\lambda_1$  as above.

2.5. *Chern-Simons line bundle on the moduli of curves.* For each  $0 \leq i \leq [g/2]$ , let  $t_i$  be a degeneration parameter of a family of stable (algebraic) curves of genus  $g$  such that the degeneration under  $t_i = 0$  is desingularized to a stable curve of genus  $g - 1$  if  $i = 0$ , and to 2 stable curves of genus  $i$  and  $g - i$  if  $i > 0$ .

PROPOSITION 2.4. *For a family of Schottky uniformized 3-manifolds  $X$  whose boundaries  $\partial X$  are Riemann surfaces of genus  $g$  degenerating as  $t_i \rightarrow 0$ , one has*

$$\left| F_1(\partial X)^6 \exp \left( 2\pi\sqrt{-1}\text{CS}^{PSL_2(\mathbb{C})} \right) \right| \sim |t_i|^{1/2} \quad (t_i \rightarrow 0)$$

which means that

$$\lim_{t_i \rightarrow 0} \left| F_1(\partial X)^6 \exp \left( 2\pi\sqrt{-1}\text{CS}^{PSL_2(\mathbb{C})} \right) \right| \cdot |t_i|^{-1/2}$$

exists and is not equal to 0.

*Proof.* First, for each  $0 \leq i \leq [g/2]$ , we consider the associated degeneration of Schottky uniformized Riemann surfaces, and show that  $F_1$  converges under this degeneration. Let  $\Gamma = \langle \gamma_1, \dots, \gamma_g \rangle$  be a Schottky group of rank  $g$ . When  $i = 0$ , let  $\gamma_g(t_0)$  be an element of  $PSL_2(\mathbb{C})$  which has fixed points same to those of  $\gamma_g$  and multiplier  $t_0$ . Then for sufficiently small  $t_0 \neq 0$ ,

$$\Gamma(t_0) = \langle \gamma_1, \dots, \gamma_{g-1}, \gamma_g(t_0) \rangle$$

is a Schottky group of rank  $g$ , and under  $t_0 \rightarrow 0$ , the associated Riemann surface  $R_{\Gamma(t_0)}$  tends to the stable curve obtained from  $R_{\langle \gamma_1, \dots, \gamma_{g-1} \rangle}$  by identifying two fixed points of  $\gamma_g$ . Furthermore, the multiplier  $q_\gamma$  of any  $\gamma \in \Gamma(t_0) - \langle \gamma_1, \dots, \gamma_{g-1} \rangle$  tends to 0 as  $t_0 \rightarrow 0$ , and hence

$$F_1(\Gamma(t_0)) \rightarrow F_1(\langle \gamma_1, \dots, \gamma_{g-1} \rangle) \sim 1 \quad (t_0 \rightarrow 0).$$

When  $i > 0$ , take two points  $a, a' \in \mathbb{P}^1(\mathbb{C})$  which are outside the Jordan curves  $C_{\pm j}$  associated with  $\gamma_j$  ( $1 \leq j \leq g$ ), and let  $\mu$  (resp.  $\mu'$ ) be elements of  $PSL_2(\mathbb{C})$  with



attractive fixed point  $a$  (resp.  $a'$ ), repulsive fixed point  $a'$  (resp.  $a$ ) and multiplier  $t_i$ . Then for sufficiently small  $t_i \neq 0$ ,

$$\Gamma(t_i) = \langle \gamma_1, \dots, \gamma_i, \mu' \gamma_{i+1} \mu, \dots, \mu' \gamma_g \mu \rangle$$

are Schottky groups of rank  $g$ , and under  $t_i \rightarrow 0$ ,  $R_{\Gamma(t_i)}$  tends to the union of  $R_{\langle \gamma_1, \dots, \gamma_i \rangle}$  and  $R_{\langle \gamma_{i+1}, \dots, \gamma_g \rangle}$  obtained by identifying  $a$  and  $a'$ . Furthermore, the multiplier  $q_\gamma$  of any  $\gamma \in \Gamma(t_i) - \langle \gamma_1, \dots, \gamma_i \rangle$  tends to 0 as  $t_i \rightarrow 0$ , and hence

$$F_1(\Gamma(t_i)) \rightarrow F_1(\langle \gamma_1, \dots, \gamma_i \rangle) \sim 1 \quad (t_i \rightarrow 0).$$

Therefore, to prove the assertion, it is enough to show that for each  $0 \leq i \leq [g/2]$ ,

$$\left| \exp \left( 2\pi \sqrt{-1} \text{CS}^{PSL_2(\mathbb{C})} \right) \right| \sim |t_i|^{1/2} \quad (t_i \rightarrow 0).$$

Since  $\text{CS}(g, S)$  is  $\mathbb{R}$ -valued,

$$\left| \exp \left( 2\pi \sqrt{-1} \text{CS}^{PSL_2(\mathbb{C})} \right) \right| = e^{\text{Vol}_R(X)/\pi} \cdot e^{-\chi(\partial X)} = e^{-S_L/(4\pi)} \cdot e^{3(g-1)}.$$

If  $i = 0$ , then [Z1, Theorem 2.4] implies that  $\exp(-S_L/(4\pi)) \sim |t_0|^{1/2}$ . If  $i > 0$ , then Theorem 2.1 implies that

$$\exp(-S_L/(4\pi)) \sim \|\varphi\|_{\bar{Q}}^{-6} \quad (t_i \rightarrow 0),$$

and by [Fr, Corollary 5.8],  $\|\varphi\|_{\bar{Q}}^{-6} \sim |t_i|^{1/2}$  as  $t_i \rightarrow 0$ . This completes the proof.  $\square$

Let  $\mathcal{M}_g$  denote the moduli stack over  $\mathbb{Z}$  of proper smooth algebraic curves of genus  $g$ . Then the associated complex orbifold  $\mathcal{M}_g(\mathbb{C})$  becomes the moduli space of Riemann surfaces of genus  $g$ . Let  $\overline{\mathcal{M}}_g$  denote the Deligne-Mumford compactification of  $\mathcal{M}_g$  as the moduli stack of stable curves of genus  $g$  (cf. [DM]). Then the complement  $\partial \mathcal{M}_g = \overline{\mathcal{M}}_g - \mathcal{M}_g$  is the union of normal crossing divisors  $\mathcal{D}_i$  defined as  $t_i = 0$  ( $0 \leq i \leq [g/2]$ ). For the universal stable curve  $\pi : \overline{\mathcal{C}}_g \rightarrow \overline{\mathcal{M}}_g$ , the Hodge line bundle  $\lambda_1 = \lambda_{g;1}$  is defined as  $\det \left( \pi_* \left( \omega_{\overline{\mathcal{C}}_g/\overline{\mathcal{M}}_g} \right) \right)$ , where  $\omega_{\overline{\mathcal{C}}_g/\overline{\mathcal{M}}_g}$  denotes the dualizing sheaf on  $\overline{\mathcal{C}}_g$  over  $\overline{\mathcal{M}}_g$ .

**THEOREM 2.5.** *The bundle  $\mathcal{L}_{\mathfrak{S}_g}^{\otimes 2}$  can be descended to a line bundle on  $\mathcal{M}_g(\mathbb{C})$  which we denote by  $\mathcal{L}_{\mathcal{M}_g(\mathbb{C})}$ . Furthermore, there exists an isomorphism*

$$\lambda_{g;1}^{\otimes 12} \xrightarrow{\sim} \mathcal{L}_{\mathcal{M}_g(\mathbb{C})}^{\otimes (-1)}$$

*which is also an isometry up to a nonzero constant.*

*Proof.* By Theorem 2.3,

$$\Phi = (F_1 \varphi)^{12} \cdot \exp \left( 2\pi \sqrt{-1} \text{CS}^{PSL_2(\mathbb{C})} \right)^2$$

gives a parallel section of the trivial bundle  $\lambda_{g;1}^{\otimes 12} \otimes \mathcal{L}_{\mathfrak{S}_g}^{\otimes 2}$  with canonical connection. Since  $\mathfrak{S}_g$  is a unramified covering of  $\mathcal{M}_g(\mathbb{C})$ ,  $\Phi$  gives a local system on  $\mathcal{M}_g(\mathbb{C})$  with coefficients in  $\mathbb{C}$  which gives rise to the monodromy representation  $\pi_1(\mathcal{M}_g(\mathbb{C})) \rightarrow \mathbb{C}^\times$ . Then by Proposition 2.4,

$$|F_1|^{12} \cdot \left| \exp \left( 2\pi\sqrt{-1}\text{CS}^{PSL_2(\mathbb{C})} \right) \right|^2 \sim |t_i| \ (t_i \rightarrow 0),$$

and hence a small loop in  $\mathcal{M}_g(\mathbb{C})$  around  $t_i = 0$  has the trivial holonomy. Since  $\pi_1(\mathcal{M}_g(\mathbb{C}))$  is the mapping class group of genus  $g$  and is generated by Dehn twists, this monodromy representation is trivial. Therefore,  $\mathcal{L}_{\mathfrak{S}_g}^{\otimes 2}$  can be descended to a hermitian holomorphic line bundle on  $\mathcal{M}_g(\mathbb{C})$  which we denote by  $\mathcal{L}_{\mathcal{M}_g(\mathbb{C})}$  such that  $\Phi$  gives an isometric isomorphism  $\lambda_{g;1}^{\otimes 12} \xrightarrow{\sim} \mathcal{L}_{\mathcal{M}_g(\mathbb{C})}^{\otimes (-1)}$  up to a nonzero constant.  $\square$

**COROLLARY 2.6.** *There exists a natural extension  $\mathcal{L}_{\overline{\mathcal{M}}_g(\mathbb{C})}$  of  $\mathcal{L}_{\mathcal{M}_g(\mathbb{C})}$  as a line bundle on  $\overline{\mathcal{M}}_g(\mathbb{C})$ . Furthermore,*

$$\lambda_{g;1}^{\otimes 12} \cong \mathcal{L}_{\overline{\mathcal{M}}_g(\mathbb{C})}^{\otimes (-1)} \otimes \mathcal{O}_{\overline{\mathcal{M}}_g(\mathbb{C})}(\partial\mathcal{M}_g)$$

*which associates  $(F_1\varphi)^{12}$  with  $\exp \left( 2\pi\sqrt{-1}\text{CS}^{PSL_2(\mathbb{C})} \right)^{-2}$  up to a nonzero constant.*

*Proof.* This follows immediately from Theorem 2.5 and its proof.  $\square$ .

### 3. Arithmetic Riemann-Roch theorem

**3.1. Riemann-Roch isomorphism.** Fix integers  $g, n \geq 0$  such that  $2g - 2 + n > 0$ , and let  $\overline{\mathcal{M}}_{g,n}$  be the moduli stack over  $\mathbb{Z}$  of stable  $n$ -pointed curves of genus  $g$  (cf. [K]). Then by definition, there exist the universal curve  $\pi : \overline{\mathcal{C}}_{g,n} \rightarrow \overline{\mathcal{M}}_{g,n}$  and the universal sections  $\sigma_j : \overline{\mathcal{M}}_{g,n} \rightarrow \overline{\mathcal{C}}_{g,n}$  ( $1 \leq j \leq n$ ). Denote by  $\mathcal{M}_{g,n}$  the open substack of  $\overline{\mathcal{M}}_{g,n}$  classifying proper smooth  $n$ -pointed curves of genus  $g$ . Then from the relative dualizing sheaf  $\omega_{\overline{\mathcal{C}}_{g,n}/\overline{\mathcal{M}}_{g,n}}$  and the complement  $\partial\mathcal{M}_{g,n} = \overline{\mathcal{M}}_{g,n} - \mathcal{M}_{g,n}$  of  $\mathcal{M}_{g,n}$ , one has the following line bundles on  $\overline{\mathcal{M}}_{g,n}$ :

$$\begin{aligned} \lambda_{g,n;k} &:= \det R\pi_* \left( \omega_{\overline{\mathcal{C}}_{g,n}/\overline{\mathcal{M}}_{g,n}}^k \left( k \sum_j \sigma_j \right) \right), \\ \psi_{g,n} &:= \bigotimes_{j=1}^n \psi_{g,n}^{(j)}; \quad \psi_{g,n}^{(j)} := \sigma_j^* \left( \omega_{\overline{\mathcal{C}}_{g,n}/\overline{\mathcal{M}}_{g,n}} \right), \\ \delta_{g,n} &:= \mathcal{O}_{\overline{\mathcal{M}}_{g,n}}(\partial\mathcal{M}_{g,n}). \end{aligned}$$

Furthermore, let  $\kappa_{g,n}$  be the line bundle on  $\overline{\mathcal{M}}_{g,n}$  defined as the following Deligne's pairing:

$$\kappa_{g,n} = \left\langle \omega_{\overline{\mathcal{C}}_{g,n}/\overline{\mathcal{M}}_{g,n}} \left( \sum_j \sigma_j \right), \omega_{\overline{\mathcal{C}}_{g,n}/\overline{\mathcal{M}}_{g,n}} \left( \sum_j \sigma_j \right) \right\rangle.$$

Using the  $k$ th residue map  $\text{Res}_{\sigma_j}^k$  given by  $\text{Res}_{\sigma_j}^k \left( \eta (dz_j/z_j)^k \right) = \eta$ , where  $z_j$  denotes a local coordinate on  $\overline{\mathcal{C}}_{g,n}$  around  $\sigma_j$ , one has

$$\lambda_{g,n;k} \cong \det \pi_* \left( \omega_{\overline{\mathcal{C}}_{g,n}/\overline{\mathcal{M}}_{g,n}}^k \left( (k-1) \sum_j \sigma_j \right) \right).$$

In particular,  $\lambda_{g,n;1} \cong \det \pi_* \left( \omega_{\overline{\mathcal{C}}_{g,n}/\overline{\mathcal{M}}_{g,n}} \right)$ . When  $n = 0$ , we delete  $n$  from these notations, and put  $\psi_{g,0} = \psi_g = \mathcal{O}_{\overline{\mathcal{M}}_g}$ . Furthermore, put  $d_k = 6k^2 - 6k + 1$  and

$$a(g, n) = (2g - 2 + n) \left( -12\zeta'_{\mathbb{Q}}(-1) + \frac{1}{2} \right),$$

where  $\zeta'_{\mathbb{Q}}(-1)$  denotes the derivative of Riemann's zeta function  $\zeta_{\mathbb{Q}}$  at  $-1$ .

**THEOREM 3.1** (arithmetic Riemann-Roch theorem [D2, Fr, GS, W]). *There exists a unique (up to a sign) isomorphism*

$$\lambda_{g,n;k}^{\otimes 12} \otimes \psi_{g,n} \otimes \delta_{g,n}^{\otimes (-1)} \cong \kappa_{g,n}^{\otimes d_k} \cdot e^{a(g,n)}$$

between the line bundles over  $\overline{\mathcal{M}}_{g,n}$  which is an isometry between the line bundles over  $\mathcal{M}_{g,n}(\mathbb{C})$  for these hermitian structure.

### 3.2. Mumford isomorphism.

**THEOREM 3.2.** *There exists a unique (up to a sign) isomorphism*

$$\mu_{g,n;k} : \lambda_{g,n;k} \xrightarrow{\sim} \lambda_{g,n;1}^{\otimes d_k} \otimes \left( \psi_{g,n} \otimes \delta_{g,n}^{\otimes (-1)} \right)^{\otimes (k^2-k)/2} \cdot \exp((k-k^2)a(g,n)/2).$$

between the line bundles over  $\overline{\mathcal{M}}_{g,n}$  which is an isometry between the line bundles over  $\mathcal{M}_{g,n}(\mathbb{C})$  for these hermitian structure. We call  $\mu_{g,n;k}$  the Mumford isomorphism (cf. [Mu2]).

*Proof.* The uniqueness follows from the properness of  $\overline{\mathcal{M}}_{g,n}$  over  $\mathbb{Z}$ . The existence is shown by Mumford [Mu2] when  $n = 0$ , namely, there exists a canonical isomorphism

$$\lambda_{g+n;k} \cong \lambda_{g+n;1}^{\otimes d_k} \otimes \delta_{g+n}^{\otimes (k-k^2)/2}.$$

Let  $\text{cl}_1 : \overline{\mathcal{M}}_{g,n} \times \overline{\mathcal{M}}_{1,1}^{\times n} \rightarrow \overline{\mathcal{M}}_{g+n}$  be the clutching morphism given by identifying the  $j$ th section  $\sigma_j$  over  $\overline{\mathcal{M}}_{g,n}$  and the unique section of the  $j$ th component of  $\overline{\mathcal{M}}_{1,1}^{\times n}$  ( $1 \leq j \leq n$ ). Then by [Fr, Corollary 3.4],

$$\begin{aligned} \text{cl}_1^* (\lambda_{g+n;k}) &\cong \lambda_{g,n;k} \boxtimes \lambda_{1,1;k}^{\boxtimes n}, \\ \text{cl}_1^* (\delta_{g+n}) &\cong \left( \delta_{g,n} \otimes \psi_{g,n}^{\otimes (-1)} \right) \boxtimes \left( \delta_{1,1} \otimes \psi_{1,1}^{\otimes (-1)} \right)^{\boxtimes n}, \end{aligned}$$

and there exists an isomorphism between  $\lambda_{g,n;k} \boxtimes \lambda_{1,1;k}^{\boxtimes n}$  and

$$\left( \lambda_{g,n;1}^{\otimes d_k} \otimes \left( \delta_{g,n} \otimes \psi_{g,n}^{\otimes(-1)} \right)^{\otimes(k-k^2)/2} \right) \boxtimes \left( \lambda_{1,1;1}^{\otimes d_k} \otimes \left( \delta_{1,1} \otimes \psi_{1,1}^{\otimes(-1)} \right)^{\otimes(k-k^2)/2} \right)^{\boxtimes n}.$$

Denote by  $\Delta : \text{Spec}(\mathbb{Z}) \rightarrow \overline{\mathcal{M}}_{1,1}^{\times n}$  the section corresponding to the  $n$  copies of the 1-pointed stable curve obtained from  $\mathbb{P}_{\mathbb{Z}}^1$  by identifying 0 and  $\infty$  with marked point 1. Then taking the pullback of the above isomorphism by  $\text{id} \times \Delta : \overline{\mathcal{M}}_{g,n} \rightarrow \overline{\mathcal{M}}_{g,n} \times \overline{\mathcal{M}}_{1,1}^{\times n}$ , we have the required isomorphism.  $\square$

**PROPOSITION 3.3.** *Let  $\text{cl}_2 : \overline{\mathcal{M}}_{g,n+2} \rightarrow \overline{\mathcal{M}}_{g+1,n}$  be the clutching morphism given by identifying the sections  $\sigma_{n+1}$  and  $\sigma_{n+2}$ . Then there exist the following canonical isomorphisms:*

$$\begin{aligned} \text{cl}_2^*(\lambda_{g+1,n;k}) &\cong \lambda_{g,n+2;k}, \\ \text{cl}_2^*(\psi_{g+1,n}) &\cong \psi_{g,n+2} \otimes \left( \psi_{g,n+2}^{(n+1)} \right)^{\otimes(-1)} \otimes \left( \psi_{g,n+2}^{(n+2)} \right)^{\otimes(-1)}, \\ \text{cl}_2^*(\delta_{g+1,n}) &\cong \delta_{g,n+2} \otimes \left( \psi_{g,n+2}^{(n+1)} \right)^{\otimes(-1)} \otimes \left( \psi_{g,n+2}^{(n+2)} \right)^{\otimes(-1)}, \end{aligned}$$

and under these isomorphisms,  $\text{cl}_2^*(\mu_{g+1,n;k}) = \pm \mu_{g,n+2;k}$ .

*Proof.* Let  $(\pi : C \rightarrow S; \sigma_1, \dots, \sigma_{n+2})$  be an  $(n+2)$ -pointed stable curve of genus  $g$ , and  $(\pi' : C' \rightarrow S; \sigma_1, \dots, \sigma_n)$  be the  $n$ -pointed stable curve of  $g+1$  obtained from  $C$  by identifying  $\sigma_{n+1}$  and  $\sigma_{n+2}$ . Then as is shown in [K, Section 1],  $\pi'_* \left( \omega_{C'/S}^k \left( k \sum_{j=1}^n \sigma_j \right) \right)$  is isomorphic to  $\text{Ker}(\rho)$ , where  $\rho : \pi_* \left( \omega_{C/S}^k \left( k \sum_{j=1}^{n+2} \sigma_j \right) \right) \rightarrow \mathcal{O}_S$  is given by

$$\rho(\eta) = \text{Res}_{\sigma_{n+1}}^k(\eta) - (-1)^k \text{Res}_{\sigma_{n+2}}^k(\eta).$$

Therefore, we have the first isomorphism. The second and third ones are shown in [K, Theorem 4.3], and hence the last identity follows from the uniqueness of the Mumford isomorphism.  $\square$

#### 4. Arithmetic Schottky uniformization

**4.1. Tate curve.** According to [Si, T], we review the Tate curve over the ring  $\mathbb{Z}[[q]]$  of integral power series of  $q$  which gives the universal elliptic curve over the ring  $\mathbb{Z}((q)) = \mathbb{Z}[[q]][1/q]$  of integral Laurent power series of  $q$ . Put

$$s_k = \sum_{n=1}^{\infty} \frac{n^k q^n}{1 - q^n}, \quad a_4(q) = -5s_3(q), \quad a_6(q) = -\frac{5s_3(q) + 7s_5(q)}{12}$$

which are seen to be in  $\mathbb{Z}[[q]]$ . Then the Tate curve  $\mathcal{E}_q$  is defined as

$$y^2 + xy = x^3 + a_4(q)x + a_6(q),$$

and is formally represented as the quotient space  $\mathbb{G}_m/\langle q \rangle$  with the origin  $o$  given by  $1 \in \mathbb{G}_m$ . Therefore,  $dz/z$  ( $z \in \mathbb{G}_m$ ) is a regular 1-form on  $\mathcal{E}_q$ , and

$$\begin{aligned} X(z) &= \sum_{n \in \mathbb{Z}} \frac{q^n z}{(1 - q^n z)^2} - 2s_1(q) \\ &= \frac{z}{(1 - z)^2} + \sum_{n=1}^{\infty} \left( \frac{q^n z}{(1 - q^n z)^2} + \frac{q^n z^{-1}}{(1 - q^n z^{-1})^2} - 2 \frac{q^n}{(1 - q^n)^2} \right), \\ Y(z) &= \sum_{n \in \mathbb{Z}} \frac{(q^n z)^2}{(1 - q^n z)^3} + s_1(q) \\ &= \frac{z^2}{(1 - z)^3} + \sum_{n=1}^{\infty} \left( \frac{(q^n z)^2}{(1 - q^n z)^3} + \frac{q^n z^{-1}}{(1 - q^n z^{-1})^3} + \frac{q^n}{(1 - q^n)^2} \right) \end{aligned}$$

are meromorphic functions on  $\mathcal{E}_q$  which have only one pole at  $o$  of order 2, 3 respectively. Hence for each positive integer  $k$ ,  $H^0 \left( \mathcal{E}_q, \omega_{\mathcal{E}_q/\mathbb{Z}[[q]]}^k(ko) \right)$  is a free  $\mathbb{Z}[[q]]$ -module of rank  $k$  generated by  $(dz/z)^k$  and  $f_i(z)(dz/z)^k$  ( $2 \leq i \leq k$ ), where  $f_i(z)$  is a meromorphic function on  $\mathcal{E}_q$  with only one pole at  $o$  of order  $i$ .

We consider the Tate curve with marked points. For variables  $t_1, \dots, t_h$ , put

$$R = \mathbb{Z} \left[ t_k, \frac{1}{t_k}, \frac{1}{t_k - 1}, \frac{1}{t_l - t_m} \mid (1 \leq k, l, m \leq h, l \neq m) \right].$$

Then each  $t_i$  gives a point on  $\mathcal{E}_q \otimes R[[q]]$  which we denote by the same symbol. When  $t_1 = 1$ , the corresponding point becomes the origin  $o$ .

PROPOSITION 4.1. *The  $R[[q]]$ -module*

$$V = H^0 \left( \mathcal{E}_q \otimes R[[q]], \Omega_{\mathcal{E}_q \otimes R[[q]]/R[[q]]}^k \left( k \sum_{j=1}^h t_j \right) \right)$$

is free of rank  $hk$ , and its basis consists of certain products of  $X(z/t_j)/t_j^2$ ,  $Y(z/t_j)/t_j^3$  times  $(dz/z)^k$ .

*Proof.* This follows from that  $\text{Res}_{z=0} (z^{k-1}(dz/z^k)) = 1$ , and that  $X(z/t_j)/t_j^2$ ,  $Y(z/t_j)/t_j^3$  have Laurent expansion at  $z = 0$  with coefficients in  $R((q))$ .  $\square$

4.2. *Arithmetic Schottky uniformization.* The arithmetic Schottky uniformization theory [I2] gives a higher genus version of the Tate curve, and its 1-forms and periods. We review this theory for the special case concerned with universal deformations of irreducible degenerate curves.

Denote by  $\Delta$  the graph with one vertex and  $g$  loops. Let  $x_{\pm 1}, \dots, x_{\pm g}$ ,  $y_1, \dots, y_g$  be variables, and put

$$A_g = \mathbb{Z} \left[ x_k, \frac{1}{x_l - x_m} \mid (k, l, m \in \{\pm 1, \dots, \pm g\}, l \neq m) \right],$$

$$\begin{aligned} A_\Delta &= A[[y_1, \dots, y_g]], \\ B_\Delta &= A_\Delta[1/y_i \ (1 \leq i \leq g)]. \end{aligned}$$

Then it is shown in [I2, Section 3] that there exists a stable curve  $C_\Delta$  of genus  $g$  over  $A_\Delta$  which satisfies the followings:

- $C_\Delta$  is a universal deformation of the universal degenerate curve with dual graph  $\Delta$  which is obtained from  $\mathbb{P}_A^1$  by identifying  $x_i$  and  $x_{-i}$  ( $1 \leq i \leq g$ ). The ideal of  $A_\Delta$  generated by  $y_1, \dots, y_g$  corresponds to the boundary  $\partial\mathcal{M}_g = \overline{\mathcal{M}}_g - \mathcal{M}_g$  of  $\overline{\mathcal{M}}_g$  via the morphism  $\text{Spec}(A_\Delta) \rightarrow \overline{\mathcal{M}}_g$  associated with  $C_\Delta$ .
- $C_\Delta$  is smooth over  $B_\Delta$ , and is Mumford uniformized (cf. [Mu1]) by the subgroup  $\Gamma_\Delta$  of  $PGL_2(B_\Delta)$  with  $g$  generators

$$\phi_i = \begin{pmatrix} x_i & x_{-i} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} y_i & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_i & x_{-i} \\ 1 & 1 \end{pmatrix}^{-1} \pmod{(B_\Delta^\times)} \quad (1 \leq i \leq g).$$

Furthermore,  $C_\Delta$  has the following universality: for a complete integrally closed noetherian local ring  $R$  with quotient field  $K$  and a Mumford curve  $C$  over  $K$  such that  $\Delta$  is the dual graph of its degenerate reduction, there is a ring homomorphism  $A_\Delta \rightarrow R$  which gives rise to  $C_\Delta \otimes_{A_\Delta} K \cong C$ .

- Let  $\Gamma = \langle \gamma_1, \dots, \gamma_g \rangle$  be a Schottky group of rank  $g$ , where each  $\gamma_i$  has the attractive (resp. repulsive) fixed points  $a_i$  (resp.  $a_{-i}$ ), and it has the multiplier  $q_i$ . Then substituting  $a_{\pm i}$  to  $x_{\pm i}$  and  $q_i$  to  $y_i$  ( $1 \leq i \leq g$ ),  $C_\Delta$  becomes the Riemann surface  $R_\Gamma$  uniformized by  $\Gamma$  if  $|q_i|$  are sufficiently small.

Actually,  $C_\Delta$  is constructed in [I2] as the quotient of a certain subspace of  $\mathbb{P}_{B_\Delta}^1$  by the action of  $\Gamma$  using the theory of formal schemes. Furthermore, as is shown in [MD] and [I1, Section 3], there exists a basis of sections

$$\begin{aligned} \omega_i &= \sum_{\phi \in \Gamma_\Delta / \langle \phi_i \rangle} \left( \frac{1}{z - \phi(x_i)} - \frac{1}{z - \phi(x_{-i})} \right) \\ &= \left( \frac{1}{z - x_i} - \frac{1}{z - x_{-i}} \right) + \sum_{\phi \in \Gamma_\Delta / \langle \phi_i \rangle - \{1\}} \left( \frac{1}{z - \phi(x_i)} - \frac{1}{z - \phi(x_{-i})} \right) + \dots \\ &\in A_\Delta \left[ \prod_{k=1}^g \frac{1}{(z - x_k)(z - x_{-k})} \right] \quad (1 \leq i \leq g) \end{aligned}$$

of the dualizing sheaf  $\omega_{C_\Delta/A_\Delta}$  on  $C_\Delta$ . These  $\omega_i$  are called *normalized* since they give rise to holomorphic 1-forms on the above  $R_\Gamma$  such that  $\oint_{C_j} \omega_i = 2\pi\sqrt{-1}\delta_{ij}$ , where  $C_j \subset \mathbb{P}^1(\mathbb{C})$  are given in 2.2. Put  $\omega = \omega_1 \wedge \dots \wedge \omega_g$ .

Let  $\Delta_{g-1}$  be the graph with one vertex and  $(g-1)$  loops, and put

$$A_{g-1} = \mathbb{Z} \left[ x_k, \frac{1}{x_l - x_m} \ (k, l, m \in \{\pm 1, \dots, \pm(g-1)\}, \ l \neq m) \right].$$

Then one has the generalized Tate curve  $C_{\Delta_{g-1}}$  of genus  $g-1$  over  $A_{g-1}[[y_1, \dots, y_{g-1}]]$ .

PROPOSITION 4.2.

- (1) *The stable curve  $C_{\Delta}|_{y_g=0}$  is obtained from  $C_{\Delta_{g-1}} \otimes_{A_{g-1}} A_g$  by identifying  $x_g = x_{-g}$ .*
- (2) *The set  $\{\omega_i|_{y_g=0}\}_{1 \leq i \leq g}$  gives a basis of sections of the dualizing sheaf on  $C_{\Delta}|_{y_g=0}$  over  $A_g[[y_1, \dots, y_{g-1}]]$  which are normalized in the sense that  $\omega_i|_{y_g=0}$  ( $i < g$ ) are normalized 1-forms on  $C_{\Delta_{g-1}}$  over  $A_{g-1}[[y_1, \dots, y_{g-1}]]$  and  $\text{Res}_{z=x_g}(\omega_g|_{y_g=0}) = 1$ .*

*Proof.* The assertion (1) follows from the universality of  $C_{\Delta}$  and  $C_{\Delta_{g-1}}$ , and (2) follows from the above universal expression of  $\omega_i$ .  $\square$

In the case when we consider the Schottky space  $\mathfrak{S}_g$  of degree  $g > 1$  as the moduli space of normalized Schottky groups, we assume that the above  $\phi_1, \dots, \phi_g$  are *normalized* by considering  $x_1, x_{-1}$  as  $0, \infty$  respectively, namely,

$$\phi_1 = \begin{pmatrix} 1 & 0 \\ 0 & y_1 \end{pmatrix} \bmod (B_{\Delta}^{\times}),$$

and by putting  $x_2 = 1$ . Then as is shown in [I2, 1.1], the corresponding generalized Tate curve  $C_{\Delta}$  is defined over  $A'_{\Delta} = A'_g[[y_1, \dots, y_g]]$ , where  $A'_g$  is obtained from  $A_g$  by deleting  $x_{-1}$  and putting  $x_1 = 0, x_2 = 1$ . Therefore, one has the associated morphism  $\text{Spec}(A'_{\Delta}) \rightarrow \overline{\mathcal{M}}_g$ .

PROPOSITION 4.3.

- (1) *The infinite products*

$$\prod_{\{\gamma\}} \prod_{m=0}^{\infty} (1 - q_{\gamma}^{1+m}), \quad (1 - q_{\gamma_1})^2 \cdots (1 - q_{\gamma_1}^{k-1})^2 (1 - q_{\gamma_2}^{k-1}) \prod_{\{\gamma\}} \prod_{m=0}^{\infty} (1 - q_{\gamma}^{k+m})$$

*have universal expression as elements of  $A'_{\Delta}$ .*

- (2) *Under  $y_2 = \dots = y_g = 0$ , these elements of  $A'_{\Delta}$  becomes  $\prod_{m=0}^{\infty} (1 - y_1^{1+m})^2$  which is primitive, i.e., not congruent to 0 modulo any rational prime.*

*Proof.* Let  $(\Gamma; \gamma_1, \dots, \gamma_g)$  be a normalized Schottky group, and for  $i = 1, \dots, g$ , put  $\gamma_{-i} = \gamma_i^{-1}$ . Then by Proposition 1.3 of [I2] and its proof, if  $\gamma \in \Gamma - \{1\}$  has the reduced expression  $\gamma_{\sigma(1)} \cdots \gamma_{\sigma(l)}$  ( $\sigma(i) \in \{\pm 1, \dots, \pm g\}$ ) such that  $\sigma(1) \neq -\sigma(l)$ , then its multiplier  $q_{\gamma}$  has universal expression as an element of  $A'_{\Delta}$  divisible by  $y_{\sigma(1)} \cdots y_{\sigma(l)}$ . Therefore, the assertion (1) holds. Since

$$(1 - y_1)^2 \cdots (1 - y_1^{k-1})^2 \prod_{m=0}^{\infty} (1 - y_1^{k+m})^2 = \prod_{m=0}^{\infty} (1 - y_1^{1+m})^2,$$

the assertion (2) holds.  $\square$

## 5. Explicit Riemann-Roch isomorphism

### 5.1. Arithmeticity of Chern-Simons invariant.

THEOREM 5.1.

- (1) *There exists an isomorphism  $\mathcal{L}_{\overline{\mathcal{M}}_g(\mathbb{C})} \cong \kappa_g^{\otimes(-1)}$  between the line bundles over  $\overline{\mathcal{M}}_g(\mathbb{C})$  which is isometric over  $\mathcal{M}_g(\mathbb{C})$ .*
- (2) *There exists a unique (up to a sign) primitive element of  $A_\Delta$  which gives a universal expression of*

$$\exp\left(4\pi\sqrt{-1}\text{CS}^{PSL_2(\mathbb{C})}\right)$$

*times a certain nonzero constant.*

*Proof.* The assertion (1) follows from Corollary 2.6 and Theorem 3.1 (in the case when  $n = 0$ ,  $k = 1$ ), and then we prove the assertion (2). By the arithmetic Schottky uniformization theory,  $\omega$  gives a trivialization of  $\lambda_{g;1}$  over  $\text{Spec}(A_\Delta)$ . Since  $\omega = (2\pi\sqrt{-1})^g \varphi$ , by the description of the isomorphism in Theorem 2.5 and Corollary 2.6, a certain multiple of the holomorphic function

$$F_1^{12} \cdot \exp\left(4\pi\sqrt{-1}\text{CS}^{PSL_2(\mathbb{C})}\right)$$

on  $\mathfrak{S}_g$  gives a trivialization over  $\text{Spec}(A_\Delta)$  of

$$\mathcal{L}_{\overline{\mathcal{M}}_g(\mathbb{C})} \otimes \mathcal{O}_{\overline{\mathcal{M}}_g}(-\partial\mathcal{M}_g) \cong \kappa_g^{\otimes(-1)} \otimes \mathcal{O}_{\overline{\mathcal{M}}_g}(-\partial\mathcal{M}_g)$$

for its  $\mathbb{Z}$ -structure induced from that on  $\kappa_g$ . As is shown in Proposition 4.3,  $F_1$  is given as an invertible element of  $A_\Delta$ , and hence the assertion (2) holds.  $\square$

*Definition.* We denote by  $\text{ACS}_g$  the element of  $A_\Delta$  given in Theorem 5.1 (2), and call it the *arithmetic universal Chern-Simons invariant*.

5.2. *Holomorphic factorization formula.* Assume that  $g > 1$ , and take an integer  $k > 1$ . Let  $(\Gamma; \gamma_1, \dots, \gamma_g)$  be a marked normalized Schottky group, and  $\mathbb{C}[z]_{2k-2}$  be the  $\mathbb{C}$ -vector space of polynomials  $f = f(z)$  of  $z$  with degree  $\leq 2k - 2$  on which  $\Gamma$  acts as

$$\gamma(f)(z) = f(\gamma(z)) \cdot \gamma'(z)^{1-k} \quad (\gamma \in \Gamma, f \in \mathbb{C}[z]_{2k-2}).$$

Take  $\xi_{1,k-1}, \xi_{2,1}, \dots, \xi_{2,2k-2}, \xi_{i,0}, \dots, \xi_{i,2k-2}$  ( $3 \leq i \leq g$ ) as elements of the Eichler cohomology group  $H^1(\Gamma, \mathbb{C}[z]_{2k-2})$  of  $\Gamma$  which are uniquely determined by the condition:

$$\xi_{i,j}(\gamma_l) = \begin{cases} \delta_{2l}(z-1)^j & (i=2), \\ \delta_{il}z^j & (i \neq 2) \end{cases}$$

for  $1 \leq l \leq g$ . Then it is shown in [MT, Section 4] that

$$\Psi_{g;k}(\psi, \xi) := \frac{1}{2\pi\sqrt{-1}} \sum_{i=1}^g \oint_{C_i} \psi \cdot \xi(\gamma_i) dz$$



for  $\psi \in H^0(R_\Gamma, \Omega_{R_\Gamma}^k)$ ,  $\xi \in H^1(\Gamma, \mathbb{C}[z]_{2k-2})$  is a non-degenerate pairing on

$$H^0(R_\Gamma, \Omega_{R_\Gamma}^k) \times H^1(\Gamma, \mathbb{C}[z]_{2k-2}).$$

Then there exists a basis

$$\{\psi_{1,k-1}, \psi_{2,1}, \dots, \psi_{2,2k-2}, \psi_{i,0}, \dots, \psi_{i,2k-2} \ (3 \leq i \leq g)\}$$

which we call the *normalized basis* of  $H^0(R_\Gamma, \Omega_{R_\Gamma}^k)$  such that  $\Psi_{g;k}(\psi_{i,j}, \xi_{l,m}) = \delta_{il} \cdot \delta_{jm}$ , where  $\delta_{ij}$  denotes Kronecker's delta.

*Remark.* Since  $-\pi \cdot \Psi_{g;k}$  is the pairing given in [MT, (4.1)],

$$\left\{ -\frac{\psi_{1,k-1}}{\pi}, -\frac{\psi_{2,1}}{\pi}, \dots, -\frac{\psi_{2,2k-2}}{\pi}, -\frac{\psi_{i,0}}{\pi}, \dots, -\frac{\psi_{i,2k-2}}{\pi} \ (3 \leq i \leq g) \right\}$$

is the *natural basis for  $n$ -differentials* defined in [MT].

In what follows, put

$$\left\{ \omega_1^{(k)}, \dots, \omega_{(2k-1)(g-1)}^{(k)} \right\} = \{ \psi_{1,k-1}, \psi_{2,1}, \dots, \psi_{2,2k-2}, \psi_{i,0}, \dots, \psi_{i,2k-2} \ (3 \leq i \leq g) \},$$

and  $\omega^{(k)} = \omega_1^{(k)} \wedge \dots \wedge \omega_{(2k-1)(g-1)}^{(k)}$ .

**THEOREM 5.2** (McIntyre-Takhtajan [MT, Theorem 2]). *Assume that  $k > 1$ .*

- (1) *There exists a holomorphic function  $F_k$  on  $\mathfrak{S}_g$  which gives an isometry between  $\lambda_{g;k}$  on  $\mathcal{M}_g(\mathbb{C})$  with Quillen metric and the holomorphic line bundle on  $\mathcal{M}_g(\mathbb{C})$  determined by the hermitian metric  $\exp(S_L/12\pi)^{d_k}$ . More precisely, there exists a positive real number  $c_{g;k}$  depending only on  $g$  and  $k$  such that*

$$\exp\left(\frac{S_L}{12\pi}\right)^{d_k} = c_{g;k} |F_k|^2 \left\| \omega^{(k)} \right\|_Q^2,$$

where  $\| \cdot \|_Q$  denotes the Quillen metric.

- (2) *On the whole Schottky space  $\mathfrak{S}_g$  classifying marked normalized Schottky groups  $(\Gamma; \gamma_1, \dots, \gamma_g)$ ,  $F_k$  is given by the absolutely convergent infinite product*

$$(1 - q_{\gamma_1})^2 \dots (1 - q_{\gamma_1}^{k-1})^2 (1 - q_{\gamma_2}^{k-1}) \prod_{\{\gamma\}} \prod_{m=0}^{\infty} (1 - q_{\gamma}^{k+m}),$$

where  $\{\gamma\}$  runs over primitive conjugacy classes in  $\Gamma - \{1\}$ .

*Remark.* As is seen in Proposition 4.3,  $F_k$  has a universal expression as an element of  $A'_\Delta$  which we denote by the same symbol.

PROPOSITION 5.3. Denote by  $\mu_{g;k}$  the Mumford isomorphism  $\mu_{g,0;k}$  given in Theorem 3.2. Then there exists a nonzero constant  $c(g;k)$  depending only on  $g, k > 1$  such that

$$c(g;k) \cdot \mu_{g;k} \left( \omega^{(k)} \right) = \frac{F_1^{d_k}}{F_k} \omega^{\otimes d_k}.$$

*Proof.* Let  $a(g)$  denote the Deligne constant  $(1-g) \left( 24\zeta'_\mathbb{Q}(-1) - 1 \right)$ . Then by Theorem 3.2,  $\mu_{g;k}$  gives rise to an isometry

$$\lambda_{g;k} \cong \lambda_{g;1}^{\otimes d_k} \cdot \exp \left( (k - k^2) a(g) / 2 \right)$$

between the metrized tautological line bundles with Quillen metric on  $\mathcal{M}_g$ . Therefore, by the formulas of Zograf and of McIntyre-Takhtajan, there exists a holomorphic function  $c(g;k)$  on the Schottky space  $\mathfrak{S}_g$  satisfying the above formula such that  $|c(g;k)|$  is a constant function. Since  $\mathfrak{S}_g$  is a connected complex manifold,  $c(g;k)$  is also a constant function.  $\square$

Let  $(\Gamma; \gamma_1, \dots, \gamma_g)$  be a marked Schottky group of rank  $g > 1$ , and  $R_\Gamma = \Omega_\Gamma / \Gamma$  be the Riemann surface uniformized by  $\Gamma$  with  $n$ -marked points given by  $s_1, \dots, s_n \in \Omega_\Gamma$ . Denote by  $\mathbb{C}[z]_d$  the  $\mathbb{C}$ -vector space of polynomials over  $\mathbb{C}$  of  $z$  with degree  $\leq d$ . For  $k > 1$ , one has a  $\mathbb{C}$ -bilinear form  $\Psi_{g,n;k}$  on

$$H^0 \left( R_\Gamma, \Omega_{R_\Gamma}^k \left( k \sum_j s_j \right) \right) \times \left( H^1(\Gamma, \mathbb{C}[z]_{2k-2}) \oplus (\mathbb{C}[z]_{k-1})^{\oplus n} \right)$$

which is defined as

$$\Psi_{g,n;k} \left( \psi(dz)^k, (\xi, (f_j)_j) \right) = \frac{1}{2\pi\sqrt{-1}} \sum_{i=1}^g \oint_{\partial D_i} \psi \cdot \xi(\gamma_i) dz + \sum_{j=1}^n \text{Res}_{s_j} (\psi \cdot f_j dz).$$

Then it is easy to see that  $\Psi_{g,n;k}$  is non-degenerate since  $\Psi_{g;k}$  is so. Let

$$\{\xi_{1,n-1}, \xi_{2,1}, \dots, \xi_{2,2k-2}, \xi_{i,0}, \dots, \xi_{i,2k-2} \mid (3 \leq l \leq g)\}$$

be the above basis of  $H^1(\Gamma, \mathbb{C}[z]_{2k-2})$ . Then this basis together with

$$\left\{ \left( z^{d_1}, \dots, z^{d_n} \right) \mid 0 \leq d_j \leq k-1 \right\}$$

give a basis of  $H^1(\Gamma, \mathbb{C}[z]_{2k-2}) \oplus (\mathbb{C}[z]_{k-1})^{\oplus n}$ . Therefore, there exists its dual basis of  $H^0 \left( R_\Gamma, \Omega_{R_\Gamma}^k \left( k \sum_j s_j \right) \right)$  for  $\Psi_{g,n;k}$  which is also called *normalized*.

### 5.3. Explicit Riemann-Roch isomorphism.

THEOREM 5.4. *The constant  $c(g; k)$  in Proposition 5.3 becomes  $\pm 1$ .*

*Proof.* As in Section 4, let  $C_\Delta$  be the generalized Tate curve over  $A'_\Delta$  of genus  $g$  which is uniformized by  $\Gamma_\Delta$ , where the generators  $\phi_1, \dots, \phi_g$  of  $\Gamma_\Delta$  is normalized as  $x_1 = 0, x_{-1} = \infty, x_2 = 1$ . As in stated in 4.1, one can also consider  $C_\Delta$  as a family of Schottky uniformized Riemann surfaces by taking  $x_{\pm i}, y_i$  as complex parameters  $a_{\pm i}, q_i$  respectively such that  $q_i$  are sufficiently small. By Proposition 4.2,  $C_\Delta|_{y_2=\dots=y_g=0}$  becomes the stable curve  $C'_\Delta$  obtained from the Tate curve  $\mathcal{E}_{y_1} = \mathbb{G}_m/\langle y_1 \rangle$  over  $A'_g[[y_1]]$  by identifying  $x_i = x_{-i}$  ( $2 \leq i \leq g$ ). Hence by the properties of the relative dualizing sheaf (cf. [K, Section 1]),  $H^0\left(C'_\Delta, \omega_{C'_\Delta/A'_g[[y_1]]}^k\right)$  becomes the subspace of

$$W = H^0\left(\mathcal{E}_{y_1}, \omega_{\mathcal{E}_{y_1}/A'_g[[y_1]]}^k \left(k \sum_{i=2}^g (x_i + x_{-i})\right)\right)$$

which consists of  $\eta \in W$  satisfying  $\text{Res}_{x_i}^k(\eta) = (-1)^k \text{Res}_{x_{-i}}^k(\eta)$ . Then by Proposition 4.1, there is a basis  $\{\eta_1, \dots, \eta_{(2k-1)(g-1)}\}$  of the  $A'_g[[y_1]]$ -module  $H^0\left(C'_\Delta, \omega_{C'_\Delta/A'_g[[y_1]]}^k\right)$  which is normalized in the sense of 5.2. Since  $k > 1$ ,  $H^1(C, \omega_C^k) = \{0\}$  for the dualizing sheaf  $\omega_C$  on a stable curve  $C$ . Therefore, the natural homomorphism

$$H^0\left(C_\Delta, \omega_{C_\Delta/A'_\Delta}^k\right) \otimes_{A'_\Delta} (A'_\Delta/(y_2, \dots, y_g)) \rightarrow H^0\left(C'_\Delta, \omega_{C'_\Delta/A'_g[[y_1]]}^k\right)$$

is surjective, and hence there exists

$$\{\theta_1, \dots, \theta_{(2k-1)(g-1)}\} \subset H^0\left(C_\Delta, \omega_{C_\Delta/A'_\Delta}^k\right)$$

such that  $\theta_l|_{y_2=\dots=y_g=0} = \eta_l$  ( $1 \leq l \leq (2k-1)(g-1)$ ).

For each  $i = 2, \dots, g$ , let

$$z_i = \frac{(x_i - x_{-i})(z - x_i)}{z - x_{-i}}, \quad z_{-i} = \frac{(x_{-i} - x_i)(z - x_{-i})}{z - x_i}$$

be the local coordinates at  $x_i, x_{-i}$  respectively such that

$$\lim_{z \rightarrow x_i} (z - x_i)/z_i = \lim_{z \rightarrow x_{-i}} (z - x_{-i})/z_{-i} = 1.$$

Then the transformation matrices of

$$(z^j)_{0 \leq j \leq l} \mapsto ((z - x_{\pm i})^j)_{0 \leq j \leq l}, \quad ((z - x_{\pm i})^j)_{0 \leq j \leq l} \mapsto \left(z_{\pm i}^j \bmod \left(z_{\pm i}^{l+1}\right)\right)_{0 \leq j \leq l}$$

have determinant 1, and hence we may replace the exterior product of the normalized basis with that of the basis of  $H^0\left(C_\Delta, \omega_{C_\Delta/A'_\Delta}^k\right)$  dual to  $\left(z_{\pm i}^j\right)_{1 \leq i \leq g}$ . Since  $C_\Delta$  is obtained as the deformation of  $\mathcal{E}_{y_1}$  by the equation  $z_i z_{-i} = -(x_i - x_{-i})^2 y_i$  ( $2 \leq i \leq g$ ),

$$\frac{(dz_i)^k}{z_i^{k+1}} \wedge \dots \wedge \frac{(dz_i)^k}{z_i^{2k-1}} = \pm ((x_i - x_{-i})^2 y_i)^{-(k^2-k)/2} \frac{(dz_{-i})^k}{z_{-i}} \wedge \dots \wedge \frac{(dz_{-i})^k}{z_{-i}^{k-1}}.$$

Let  $\{\xi_1, \dots, \xi_{(2k-1)(g-1)}\}$  be the natural basis

$$\{\xi_{1,k-1}, \xi_{2,1}, \dots, \xi_{2,2k-2}, \xi_{i,0}, \dots, \xi_{i,2k-2} \ (3 \leq i \leq g)\}$$

given in 5.2. Then

$$\prod_{i=2}^g ((x_i - x_{-i})^2 y_i)^{(k^2-k)/2} \det(\Psi_{g;k}(\theta_l, \xi_m))_{l,m}$$

is a holomorphic function of  $y_2, \dots, y_g$  around the point defined as  $y_2 = \dots = y_g = 0$ . Furthermore, under  $y_i \rightarrow 0$  ( $2 \leq i \leq g$ ),  $\frac{1}{2\pi\sqrt{-1}} \oint_{C_{\pm i}} \rightarrow \text{Res}_{x_{\pm i}}$ , and hence

$$\prod_{i=2}^g ((x_i - x_{-i})^2 y_i)^{(k^2-k)/2} \det(\Psi_{g;k}(\theta_l, \xi_m))_{l,m} \rightarrow \pm 1.$$

By definition, the normalized basis  $\{\omega_l^{(k)}\}$  of holomorphic  $k$ -forms on  $C_\Delta$  satisfies

$$\det(\Psi_{g;k}(\omega_l^{(k)}, \xi_m))_{l,m} = 1$$

which implies that

$$\prod_{i=2}^g ((x_i - x_{-i})^2 y_i)^{-(k^2-k)/2} \bigwedge_{l=1}^{(2k-1)(g-1)} \omega_l^{(k)}$$

gives a holomorphic section of  $\omega_{C_\Delta}^k$  around  $y_2 = \dots = y_g = 0$ , and it becomes  $\pm \bigwedge_{l=1}^{(2k-1)(g-1)} \eta_l$  at  $y_2 = \dots = y_g = 0$ . Then by applying Proposition 3.3, Theorem 3.2 to Proposition 5.3, and by using Propositions 4.2, 4.3, we have

$$c(g; k) \cdot \mu_{1,2g-2;k} \left( \bigwedge_{l=1}^{(2k-1)(g-1)} \eta_l \right) = \pm \prod_{m=1}^{\infty} (1 - y_1^m)^{2(d_k-1)} \cdot \prod_{i=2}^g (x_i - x_{-i})^{-(k^2-k)}$$

under the trivialization by the basis of  $H^0(C'_\Delta, \omega_{C'_\Delta/A'_g[[y_1]]}^k)$  and  $d(z - x_{\pm i})$  ( $2 \leq i \leq g$ ). Since  $\mu_{1,2g-2;k} \left( \bigwedge_{l=1}^{(2k-1)(g-1)} \eta_l \right)$  and the right hand side is primitive, the constant  $c(g; k)$  is seen to be  $\pm 1$ .  $\square$

**THEOREM 5.5.** *The Riemann-Roch isomorphism*

$$\lambda_{g;k}^{\otimes 12} \otimes \delta_g^{\otimes (-1)} \xrightarrow{\sim} \kappa_g^{\otimes d_k}$$

given by Theorem 3.1 maps  $(F_k \omega^{(k)})^{12}$  to  $\pm (\text{ACS}_g)^{-d_k}$ .

*Proof.* First, we prove the assertion when  $k = 1$ . By the definition of  $\text{ACS}_g$ , the image of  $(F_1 \omega)^{12}$  is a constant multiple of  $(\text{ACS}_g)^{-1}$ , and by the arithmetic Schottky

uniformization theory, the both are primitive. Therefore, they are equal up to a sign. Second, we assume that  $k > 1$ . Since the norm of  $\text{ACS}_g$  is a constant multiple of  $\exp(-S_L/(2\pi))$ , Theorem 5.1 implies that the image of  $(F_k \omega^{(k)})^{12}$  is a certain multiple of  $(\text{ACS}_g)^{-d_k}$ . Since  $F_1$  and  $F_k$  are expressed as primitive elements of  $A'_\Delta$ , by Theorem 5.4,  $\omega^{(k)}$  gives a trivialization of  $\lambda_{g;k}$  and is primitive. Therefore, the image of  $(F_k \omega^{(k)})^{12}$  is also primitive. This completes the proof.  $\square$

**THEOREM 5.6.** *Let  $\{\psi_l\}$  be the normalized basis of  $H^0(R_\Gamma, \Omega_{R_\Gamma}^k(k \sum_j s_j))$  for Schottky uniformized Riemann surfaces  $R_\Gamma$  with marked points  $s_1, \dots, s_n$ . Then*

$$\mu_{g,n;k} \left( \bigwedge_l \psi_l \right) = \pm \frac{F_1^{d_k}}{F_k} \omega^{\otimes d_k} \otimes \bigotimes_{j=1}^n d(z - s_j)^{\otimes (k^2 - k)/2}.$$

*Proof.* Let  $(\pi : C \rightarrow S; \sigma_1, \dots, \sigma_n)$  be an  $n$ -pointed stable curve of genus  $g > 1$ . Then for non-negative integers  $l_1, \dots, l_n \leq k$ , the  $l_i$ th residue map gives the exact sequence

$$\begin{aligned} 0 &\rightarrow \pi_* \left( \omega_{C/S}^k \left( \sum_{j=1}^n l_j \sigma_j \right) \right) \rightarrow \pi_* \left( \omega_{C/S}^k \left( \sum_{j \neq i} l_j \sigma_j + (l_i + 1) \sigma_i \right) \right) \\ &\rightarrow \mathcal{O}_S(dz_{\sigma_i})^{k-l_i-1} \rightarrow 0, \end{aligned}$$

where  $dz_{\sigma_i}$  denotes the local coordinate around  $\sigma_i$ . Therefore, by induction,

$$\det \pi_* \left( \omega_{C/S}^k \left( k \sum_{j=1}^n \sigma_j \right) \right) \cong \det \pi_* \left( \omega_{C/S}^k \right) \cdot \prod_{j=1}^n (dz_{\sigma_j})^{(k^2 - k)/2}.$$

If  $C$  consists of Schottky uniformized Riemann surfaces with marked points  $s_1, \dots, s_n$ , then the residue map sends the normalized basis to  $\{d(z - s_i)^{k-l_i-1}\}$ , and hence the assertion follows from Theorem 5.4.  $\square$

#### 5.4. Completed formulae of Zograf and McIntyre-Takhtajan.

**THEOREM 5.7.** *Let  $c_g$  be the constant given in Theorem 2.1. Then*

$$c_g = (2\pi)^{2g} \cdot \exp \left( \frac{(g-1) \left( 24\zeta'_\mathbb{Q}(-1) + 1 \right)}{6} \right).$$

*Proof.* Let  $\mathcal{C}$  be an algebraic curve of genus  $g$  over a scheme  $S$ , and represent  $\Omega_{\mathcal{C}/S}$  as  $\mathcal{O}_{\mathcal{C}}(\mathcal{D})$  and  $\mathcal{O}_{\mathcal{C}}(\mathcal{D}')$  such that the supports of  $\mathcal{D}$ ,  $\mathcal{D}'$  are disjoint. Then by definition,  $\langle \mathbf{1}, \mathbf{1} \rangle$  is a section of the line bundle  $\langle \Omega_{\mathcal{C}/S}, \Omega_{\mathcal{C}/S} \rangle$  on  $S$  given by the Deligne pairing. Denote by  $S_A[\log \rho]$  the Liouville action defined in [A, Definition 4.1]. Then by the Gauss-Bonnet theorem,  $S_L = 2\pi S_A[\log \rho]$  when  $\rho$  is the hyperbolic metric

with constant curvature  $-1$ . Since the tangent line bundle  $\mathcal{T}_{\mathcal{C}/S}$  of  $\mathcal{C}/S$  is  $\Omega_{\mathcal{C}/S}^{\otimes(-1)}$ ,  $\langle \mathcal{T}_{\mathcal{C}/S}, \mathcal{T}_{\mathcal{C}/S} \rangle = \langle \Omega_{\mathcal{C}/S}, \Omega_{\mathcal{C}/S} \rangle$ . Furthermore, if  $\mathcal{C}/S$  is a family of Riemann surfaces, then [A, Corollary 5.2] implies that

$$\exp(S_L/(2\pi)) = \|\langle \mathbf{1}, \mathbf{1} \rangle\| \cdot e^{2g-2}.$$

Let  $\omega_i$  ( $1 \leq i \leq g$ ) be as in 4.2 which becomes the normalized basis of holomorphic 1-forms on Schottky uniformized Riemann surfaces  $R_\Gamma$ , and put  $\omega = \omega_1 \wedge \cdots \wedge \omega_g$ . Then  $\int_{\alpha_i} \omega_j = 2\pi\sqrt{-1}\delta_{ij}$ , and hence

$$\frac{\det \operatorname{Im}(\tau)}{\det' \Delta_h} = \frac{\|\omega/(2\pi\sqrt{-1})^g\|_{\mathbb{H}}^2}{\det' \Delta_h} = \frac{\|\omega\|_{\mathbb{Q}}^2}{(2\pi)^{2g}}.$$

Furthermore, by Theorem 2.1, there exists an isometry  $\lambda_1^{\otimes 12} \xrightarrow{\sim} \langle \Omega_{\mathcal{C}/S}, \Omega_{\mathcal{C}/S} \rangle$  which sends  $(F_1\omega)^{12}$  to  $\langle \mathbf{1}, \mathbf{1} \rangle$  up to a nonzero constant. Since  $\omega$  and  $\langle \mathbf{1}, \mathbf{1} \rangle$  give primitive local sections of  $\lambda_{g;1}$  and  $\kappa_g$  respectively, the Riemann-Roch isomorphism sends  $(F_1\omega)^{12}$  to  $\pm \langle \mathbf{1}, \mathbf{1} \rangle$ . Therefore, by Theorem 3.1,

$$\begin{aligned} \left( \frac{\det \operatorname{Im}(\tau)}{\det' \Delta_h} \right)^6 &= \|\omega\|_{\mathbb{Q}}^{12} \cdot (2\pi)^{-12g} \\ &= \|\langle \mathbf{1}, \mathbf{1} \rangle\| \cdot |F_1|^{-12} \cdot e^{a(g)} \cdot (2\pi)^{-12g} \\ &= \exp\left(\frac{S_L}{2\pi}\right) \cdot |F_1|^{-12} \cdot \exp\left((1-g)(24\zeta'_{\mathbb{Q}}(-1) + 1)\right) \cdot (2\pi)^{-12g}. \end{aligned}$$

Comparing this equality with the definition of  $c_g$  given in Theorem 2.1, we have

$$c_g = (2\pi)^{2g} \cdot \exp\left(\frac{(g-1)(24\zeta'_{\mathbb{Q}}(-1) + 1)}{6}\right)$$

which completes the proof.  $\square$

**THEOREM 5.8.** *The constant  $c_{g;k}$  in Theorem 5.2 is determined as*

$$c_{g;k} = \exp\left(\frac{(g-1)(24\zeta'_{\mathbb{Q}}(-1) + 2d_k - 1)}{6}\right).$$

*Proof.* By Theorem 3.1,

$$\begin{aligned} \left\| \omega^{(k)} \right\|_{\mathbb{Q}}^{12} &= \|\langle \mathbf{1}, \mathbf{1} \rangle\|^{d_k} \cdot |F_k|^{-12} \cdot e^{a(g)} \\ &= \exp\left(\frac{S_L}{2\pi}\right)^{d_k} \cdot |F_k|^{-12} \cdot \exp\left((1-g)(24\zeta'_{\mathbb{Q}}(-1) + 2d_k - 1)\right) \end{aligned}$$

which completes the proof.  $\square$

## 6. Rationality of Ruelle zeta values

6.1. *Mumford isomorphism and discriminant.* Assume that  $g > 1$ , let  $(\Gamma; \gamma_1, \dots, \gamma_g)$  be a marked normalized Schottky group, and  $F_k(\Gamma; \gamma_1, \dots, \gamma_g)$  be the value of  $F_k$  at the corresponding point on  $\mathfrak{S}_g$ . Recall that  $R_\Gamma$  denotes the Riemann surface uniformized by  $\Gamma$  which is equivalently the algebraic curve over  $\mathbb{C}$  associated with  $\Gamma$ . Assume that there exists a sub  $\mathbb{Z}$ -algebra  $R$  of  $\mathbb{C}$  over which  $R_\Gamma$  has a model  $C_\Gamma$  as a stable curve, and that there exist  $R$ -basis  $\{u_1, \dots, u_g\}$  of  $H^0(C_\Gamma, \omega_{C_\Gamma/R})$  and  $\{v_1, \dots, v_{(2k-1)(g-1)}\}$  of  $H^0(C_\Gamma, \omega_{C_\Gamma/R}^k)$  for  $k > 1$ . Then their periods are defined as

$$\Omega_1 = \det \left( \frac{1}{2\pi\sqrt{-1}} \oint_{C_i} u_j \right)_{1 \leq i, j \leq g},$$

and

$$\Omega_k = \det \left( \frac{1}{2\pi\sqrt{-1}} \sum_{i=1}^g \oint_{C_i} v_l \cdot \xi_m(\gamma_i) \right)_{1 \leq l, m \leq (2k-1)(g-1)},$$

where  $C_1, \dots, C_g \subset \mathbb{P}^1(\mathbb{C})$  are given in 2.2, and  $\{\xi_1, \dots, \xi_{(2k-1)(g-1)}\}$  is the basis

$$\{\xi_{1,k-1}, \xi_{2,1}, \dots, \xi_{2,2k-2}, \xi_{i,0}, \dots, \xi_{i,2k-2} \ (3 \leq i \leq g)\}$$

of  $H^1(\Gamma, \mathbb{C}[z]_{2k-2})$  given in 5.2.

Following [S], let  $D(C_\Gamma)$  denote the discriminant ideal associated with  $C_\Gamma$ , which is defined as the ideal of  $R$  corresponding to the boundary  $\partial\mathcal{M}_g = \overline{\mathcal{M}}_g - \mathcal{M}_g$  of  $\overline{\mathcal{M}}_g$  via the morphism  $\text{Spec}(R) \rightarrow \overline{\mathcal{M}}_g$  associated with  $C_\Gamma$ . Then we consider the rationality of

$$\frac{F_k(\Gamma; \gamma_1, \dots, \gamma_g)}{\Omega_k},$$

and its relation with  $D(C_\Gamma)$ .

**THEOREM 6.1.**

(1) *Under the above notations, the ratio*

$$\frac{(F_1(\Gamma; \gamma_1, \dots, \gamma_g)/\Omega_1)^{d_k}}{F_k(\Gamma; \gamma_1, \dots, \gamma_g)/\Omega_k}$$

*belongs to the power  $D(C_\Gamma)^{(k^2-k)/2}$  of  $D(C_\Gamma)$ .*

(2) *If  $R$  is a discrete valuation ring and  $d(C_\Gamma)$  is a generator of  $D(C_\Gamma)$ , then*

$$\frac{(F_1(\Gamma; \gamma_1, \dots, \gamma_g)/\Omega_1)^{d_k}}{F_k(\Gamma; \gamma_1, \dots, \gamma_g)/\Omega_k} \in d(C_\Gamma)^{(k^2-k)/2} \cdot R^\times,$$

*where  $R^\times$  denotes the unit group of  $R$ .*

(3) If  $C_\Gamma$  is smooth over  $R$ , then

$$\frac{(F_1(\Gamma; \gamma_1, \dots, \gamma_g)/\Omega_1)^{d_k}}{F_k(\Gamma; \gamma_1, \dots, \gamma_g)/\Omega_k} \in R^\times.$$

*Proof.* By Theorem 3.2,

$$\frac{(F_1(\Gamma; \gamma_1, \dots, \gamma_g)/\Omega_1)^{d_k}}{F_k(\Gamma; \gamma_1, \dots, \gamma_g)/\Omega_k}$$

gives (up to a sign) the evaluation of the Mumford isomorphism  $\mu_{g,0;k}$  on  $C_\Gamma$  under the trivializations of  $\lambda_{g;1}$  and  $\lambda_{g;k}$  by  $u_1 \wedge \dots \wedge u_g$  and  $v_1 \wedge \dots \wedge v_{(2k-1)(g-1)}$  respectively. Hence the assertions follows from Theorem 5.4.  $\square$

**6.2. Rationality of Ruelle zeta values.** Let  $\Gamma$  be a Schottky group, and  $X_\Gamma = \mathbb{H}^3/\Gamma$  be the hyperbolic 3-manifold uniformized by  $\Gamma$ . Then for each  $\gamma \in \Gamma$ ,  $-\log |q_\gamma|$  is the length of the closed geodesic on  $X_\Gamma$  corresponding to  $\gamma$ , and hence the Ruelle zeta function of  $X_\Gamma$  becomes

$$Z_\Gamma(s) = \prod_{\{\gamma\}} (1 - |q_\gamma|^s)^{-1},$$

where  $\{\gamma\}$  runs over primitive conjugacy classes of  $\Gamma$ . It is known (cf. [MT, 5.2]) that  $Z_\Gamma(s)$  is absolutely convergent if  $\text{Re}(s) \geq 2$ . We assume that a marked normalized Schottky group  $(\Gamma; \gamma_1, \dots, \gamma_g)$  is contained in  $PSL_2(\mathbb{R})$ , and apply Theorem 6.1 to showing the rationality of the *modified* Ruelle zeta values

$$\tilde{Z}_\Gamma(k) = Z_\Gamma(k) \frac{(1 - q_{\gamma_1}^k)^2 (1 - q_{\gamma_2}^k)}{(1 - q_{\gamma_2}^{k-1})}$$

for integers  $k > 1$ .

**THEOREM 6.2.** *Assume that a Schottky group  $\Gamma$  is contained in  $PSL_2(\mathbb{R})$ , and that  $R_\Gamma$  has a model as a stable curve  $C_\Gamma$  over a sub  $\mathbb{Z}$ -algebra  $R$  of  $\mathbb{C}$ . Let  $k$  is an integer  $> 1$ , and put*

$$c(\Gamma) = \frac{F_1(\Gamma; \gamma_1, \dots, \gamma_g)}{\Omega_1}.$$

(1) If  $K$  denotes the quotient field of  $R$ , then

$$\tilde{Z}_\Gamma(k) \in \frac{\Omega_{k+1}}{\Omega_k} \cdot c(\Gamma)^{12k} \cdot K^\times.$$

(2) If  $R$  is a discrete valuation ring, then

$$\tilde{Z}_\Gamma(k) \in \frac{\Omega_{k+1}}{\Omega_k} \cdot c(\Gamma)^{12k} \cdot d(C_\Gamma)^{-k} \cdot R^\times.$$



(3) If  $C_\Gamma$  is smooth over  $R$ , then

$$\tilde{Z}_\Gamma(k) \in \frac{\Omega_{k+1}}{\Omega_k} \cdot c(\Gamma)^{12k} \cdot R^\times.$$

*Proof.* By assumption,  $q_\gamma = |q_\gamma|$  for any  $\gamma \in \Gamma - \{1\}$ , and hence

$$\tilde{Z}_\Gamma(k) = \frac{F_{k+1}(\Gamma; \gamma_1, \dots, \gamma_g)}{F_k(\Gamma; \gamma_1, \dots, \gamma_g)}.$$

Therefore, the assertions follows from Theorem 6.1.  $\square$

**COROLLARY 6.3.** *Let the notation and assumption be as in Theorem 6.2, and assume that  $R$  is a Dedekind ring with quotient field  $K$ . For  $j \in \{1, k, k+1\}$ , let  $\{v_i^{(j)}\}_i$  be a  $K$ -basis of  $H^0(C_\Gamma, \omega_{C_\Gamma/R}^j) \otimes_R K$ , and denote by  $\Omega_j$  the period of this basis.*

(1) *There exists an element  $r(C_\Gamma)$  of  $K^\times$  such that*

$$\tilde{Z}_\Gamma(k) \in \frac{\Omega_{k+1}}{\Omega_k} \cdot c(\Gamma)^{12k} \cdot r(C_\Gamma) \cdot R^\times.$$

(2) *For each prime ideal  $P$  of  $R$ ,  $\text{ord}_P(r(C_\Gamma))$  is given by*

$$k \cdot \text{ord}_P(d(C_\Gamma)) + 12k \cdot \text{ord}_P\left(\bigwedge v_i^{(1)}\right) + \text{ord}_P\left(\bigwedge v_i^{(k)}\right) - \text{ord}_P\left(\bigwedge v_i^{(k+1)}\right),$$

*where  $\text{ord}_P\left(\bigwedge v_i^{(j)}\right)$  denotes the order at  $P$  of the fractional ideal of  $R_P$  generated by the exterior product of  $\{v_i^{(j)}\}_i$  under an identification*

$$R_P = \det\left(H^0\left(C_\Gamma, \omega_{C_\Gamma/R}^j\right)\right) \otimes_R R_P$$

*as  $R_P$ -modules.*

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